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On local existence of the Vlasov-Fokker-Planck equation in a 2D anisotropic space

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Abstract

We are concerned with local existence of the Vlasov-Fokker-Planck equation in a 2*D* anisotropic space $L_x^p L_v^1$ in a bounded domain with respect to the space variable. The energy method is used to construct the result and the Hardy inequality is used to estimate the electric field.

Keywords: Vlasov-Fokker-Planck equation; anisotropic space; Hardy inequality; energy method

1 Introduction

Through this paper, as in the title, when anisotropic space is mentioned, we mean that the space variable x and the velocity variable v have different regularities, such as integrability and differentiability. Firstly, we would like to give a review which has close relationship with our paper. As to a classical solution, it is worthy to mention the paper of Yang and Yu [1]. They constructed the global-in-time classical solutions to the Vlasov-Maxwell-Fokker-Planck system near Maxwellian using an approach by combining the compensating function and energy method. For the system of (1.1) in the whole space, there are many results about weak solutions. We refer to them as follows. Triolo [2] obtained the global (local) existence in 1 and 2-dimension (3-dimension) for the Cauchy problem of the Vlasov-Poisson-Fokker-Planck equation under the initial value $f_0 \in L^1 \cap L^\infty \cap C^0(\mathbb{R}^n \times \mathbb{R}^n)$, $n = 1, 2, 3$. Neunzert and some other authors [3] considered the modified Vlasov-Fokker-Planck equation problem, under the condition that the initial value $f_0 \in L^\infty(\mathbb{R}^2 \times \mathbb{R}^2)$ has compact support in v, they got a solution which is a probability measure. Zheng and Majda [4] gave the existence of a global weak solution of the Vlasov-Fokker-Planck system under the measure-valued initial data. Degond [5] obtained the existence of solution which has the same regularity as the initial value

$$f_0 \in W^{1,1}(\mathbb{R}^{2n}), \qquad (1 + |v|^2)(|f_0| + |Df_0|) \in L^\infty.$$

The author required that f_0 has the same index of differentiability and integrability with respect to variables x and v, *i.e.*, they are isotropic. Comparing with [5], we do not need f_0, Df_0 to be bounded, continuous to offset, we cannot obtain the global existence even in a 2D space because of the difficulty to get the Gronwall inequality which can be solved in

$t \in [0, \infty)$. Roughly speaking, the essential difficult point lies in that $\|\nabla_x E^n \cdot \nabla_v f^{n+1}\|_{L_x^p L_v^1}$ is in proportion to $\|f\|^2_{L_x^p L_v^1}$, which is super-linear.

In this paper, we consider the initial boundary problem with 0 boundary value (see (1.1)). The problem about general boundary value [6, 7] is so tough that we are going to consider it in the next paper. Moreover, we focus on the case in which the initial value f_0 has different regularities with respect to the x and v variables, *i.e.*, they have different integral and differential indices. An anisotropic space is natural since the variables x and v need not to have the same regularity. For instance, Strain [8] considered the anisotropic space $\dot{B}_2^{-\varrho,\infty} L_v^2$ when he studied the optimal decay rate of the solution to the hard and soft potential Boltzmann equation.

Before narrating the main theorem, we would like to introduce some notations.

Notation

(i) $f(x, v, t)$, $x \in \Omega \subset \mathbb{R}^2$ with $\partial\Omega \in C^1$, $v \in \mathbb{R}^2$, $t > 0$, the distribution function of the particles, where Ω is a bound open set with a C^1 boundary (for Definition see p.626 of [9]).

(ii) $L_x^p L_v^1$ is a space in which the elements are given the norm $(\int_\Omega \|f\|^p_{L_v^1}\,dx)^{\frac{1}{p}}$, $1 \le p < \infty$, here L_v^1 is the Lebesgue space with the norm given by $\|f\|_{L_v^1} := \int_{\mathbb{R}^2} |f|\,dv$.

(iii) $L^2([0, T] \times \Omega_x, H^1(\mathbb{R}_v^2))$ is a space given the norm $(\int_{[0,T]\times\Omega_x} \|f\|^2_{H_v^1}\,dt\,dx)^{\frac{1}{2}}$, here H_v^1 is the Sobolev space with respect to v.

Now we are ready to state our main theorem.

In the theorem, we denote respectively by $E(x, t)$, $\rho(x, t)$ the electric field and the electric charge. $\sigma > 0$ is a diffusion coefficient which is very small in physical situations.

Theorem 1.1 *Suppose* $f_0 \in L^2(\Omega \times \mathbb{R}_v^2)$ *and* $\|f_0\|_{L_v^1}|_{\partial\Omega} = 0$. *Also, we assume that* f_0 *is nonnegative and*

$$\|f_0\|_{L_x^p L_v^1}, \|\nabla_v f_0\|_{L_x^p L_v^1}, \|\nabla_x f_0\|_{L_x^p L_v^1} \le const.$$

Here $p = \frac{4}{3}$. *Then the Vlasov-Fokker-Planck system with bounded domain*

$$\begin{cases} \frac{\partial f}{\partial t} + v \cdot \nabla_x f + E|_\Omega \cdot \nabla_v f - \sigma \Delta_v f = 0; \\ E(x, t) = \int_{\mathbb{R}^2} \frac{x-y}{|x-y|^2} \rho(y, t)\,dy; \\ \rho(x, t) = \int_{\mathbb{R}^2} f(x, v, t)\,dv; \\ f(x, v, 0) = f_0(x, v), \qquad \|f(x, v, t)\|_{L_v^1}|_{\partial\Omega} = 0 \quad \text{for } t > 0 \end{cases} \tag{1.1}$$

admits a solution in the interval $[0, T)$, *i.e.*, $f(x, v, t) \in L^\infty_{\mathrm{loc}}([0, T), L_x^p L_v^1)$. *Here we mean the boundary value in the sense of trace* (*see p.*259 *of* [9]). *Moreover, we have that* $\nabla_x f, \nabla_v f \in L^\infty_{\mathrm{loc}}([0, T), L_x^p L_v^1)$. *The solution is unique.*

The arrangement of this paper is the following. In Section 2, we cite or prove some lemmas which will be used in the proof of Theorem 1.1. In Section 3, we give the proof of the main theorem in several steps.

2 Some lemmas

Consider the linear equation, *i.e.*,

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f + E|_\Omega \cdot \nabla_v f - \sigma \Delta_v f = U; \qquad f(x,v,0) = f_0(x,v). \tag{2.1}$$

Next we introduce some lemmas, which are modified versions of the lemmas in Appendix A in [5] and will be used in the proof of Theorem 1.1 in Section 3. We only point out the differences, the details can be found in the reference.

Lemma 2.1 *Assume*

$$\begin{aligned} &f_0 \in L^2(\Omega \times \mathbb{R}^2); \qquad U \in L^2([0,T] \times \Omega_x, H^{-1}(\mathbb{R}^n_v)); \\ &E(x,t) \in L^\infty([0,T], L^p(\mathbb{R}^2_x)), \end{aligned} \tag{2.2}$$

where $p > 2$. *Then equation* (2.1) *has a unique solution* f *in a class of functions* Y *defined according to*

$$Y = \left\{ f \in L^2([0,T] \times \Omega_x, H^1(\mathbb{R}^2_v)), \forall \textit{ fixed } x \in \Omega, \frac{\partial f}{\partial t} + v \cdot \nabla_x \in L^2([0,T], H^{-1}(\mathbb{R}^2_v)) \right\}.$$

Proof Since $E(x,t) \in L^\infty_t L^p_x$, $\iint \nabla_v \varphi \cdot E\varphi \, dx \, dv$ exists for any $\varphi \in \mathcal{D}$, which implies $\int \nabla_v \varphi \cdot E\varphi \, dv = 0$.

On the other hand, since $f \in L^2([0,T] \times \Omega_x, H^1(\mathbb{R}^2_v))$, we have $\forall x \in \Omega$, $\nabla_v f \in L^2([0,T], H^{-1}(\mathbb{R}^2_v))$, which implies $\forall x \in \Omega$,

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f = U - E \cdot \nabla_v f + \sigma \Delta_v f \in L^2([0,T], H^{-1}(\mathbb{R}^2_v)).$$

The other processes are similar. □

Remark 2.2 Similar to Lemma A.1 on p.535 of [5], we mean the initial condition that for any $u \in Y$, u admits trace value $u(x,v,0) \in L^2_v$ for a.e. $x \in \Omega$.

Lemma 2.3 *Assume in addition* $f_0 \in L^p_x L^1_v$; $U \in L^1_t L^p_x L^1_v$, *then the solution defined in Lemma* 2.1 *belongs to* $L^\infty([0,T], L^p_x L^1_v)$ *and satisfies for* $t \in [0,T]$ *a.e.*,

$$\|f(t)\|_{L^p_x L^1_v} \le \|f_0\|_{L^p_x L^1_v} + \int_0^t \|U(s)\|_{L^p_x L^1_v} \, ds. \tag{2.3}$$

Proof Using quite a similar method to that on p.540 of [5], we get for a.e. x,

$$\|f(t,x)\|_{L^1_v} \le \|f_0\|_{L^1_v} + \int_0^t \|U(s)\|_{L^1_v} \, ds.$$

Taking the L^p norm with respect to x yields the desired result. □

3 Proof of the main theorem

After the above preparation, we are in a position to prove our main theorem.

Proof To prove the theorem, we split the process into several steps. The first step is to construct the approximating solution sequence; in the second step, we prove the regularity of the solution we have obtained; and in the last step, we prove the uniqueness. To get the existence of a weak solution in the space L_x^p, it is natural to require f^n to be a Cauchy sequence in the strong topology of L_x^p and $E(t,x)$ to be bounded in $L_x^{p'}$, here $\frac{1}{p}+\frac{1}{p'}=1$. To estimate the $L^{p'}$ norm of $E(t,x)$, we will use the Hardy inequality.

Step 1: We construct an iterative solution sequence to approximate the solution of the original equation.

$$\begin{cases} \frac{\partial f^{n+1}}{\partial t}+v\cdot\nabla_x f^{n+1}+E^n|_\Omega\cdot\nabla_v f^{n+1}-\sigma\Delta_v f^{n+1}=0; \\ E^n(x,t)=\int_{\mathbb{R}^2}\frac{x-y}{|x-y|^2}\rho^n(y,t)\,dy; \\ \rho^n(x,t)=\int_{\mathbb{R}^2}f^n(x,v,t)\,dv; \\ f^{n+1}(x,v,0)=f_0(x,v); \\ \|f^{n+1}\|_{L_v^1}|_{\partial\Omega}=0. \end{cases} \tag{3.1}$$

Firstly, on the one hand, to guarantee the weak convergence, we need the weak convergence of E^n in $L^{p'}$, which in turn can be deduced by the boundedness of E^n in $L^{p'}$. By the Hardy inequality [10–12], we have

$$\|E^n\|_{L^{p'}}\le\left\|\frac{1}{|x|}\underset{x}{*}\rho^n\right\|_{L^{p'}}\le C\|f^n\|_{L_x^pL_v^1},$$

where

$$\frac{1}{p}+\frac{1}{2}=1+\frac{1}{p'}. \tag{3.2}$$

On the other hand, by Lemma 2.1 and Lemma 2.3, let $f^0=0$, we obtain a unique solution f^{n+1} to (3.1). To pass to the limit in the system of (1.1), we need a strong convergence. In order to get this, we compute the Cauchy sequence as follows:

$$\begin{aligned} &\left\|f^{n+1}(x,v,t)-f^n(x,v,t)\right\|_{L_x^pL_v^1} \\ &\quad\le\int_0^t\left\|\left(E^n(x,s)-E^{n-1}(x,s)\right)\left\|\nabla_v f^n\right\|_{L_v^1}\right\|_{L_x^p}ds \\ &\quad\le\int_0^t\left\|\frac{1}{|x|}\underset{x}{*}\left|\rho^n-\rho^{n-1}\right|\left\|\nabla_v f^n\right\|_{L_v^1}\right\|_{L_x^p}ds \\ &\quad\le\int_0^t\int_{\mathbb{R}_y^2}\left|\rho^n(y,s)-\rho^{n-1}(y,s)\right|dy \\ &\qquad\cdot\left(\int_{\mathbb{R}_x^2}\left(\frac{1}{|x-y|^{n-1}}\left\|\nabla_v f^n(x,v,s)\right\|_{L_v^1}\right)^p dx\right)^{\frac{1}{p}}ds \\ &\quad\le\int_0^t\left\|\rho^n(\cdot,s)-\rho^{n-1}(\cdot,s)\right\|_{L_x^p}\left\|\left(\frac{1}{|x|^p}\underset{x}{*}\left\|\nabla_v f^n(x,v,s)\right\|_{L_v^1}^p\right)^{\frac{1}{p}}\right\|_{L_x^{p'}}ds. \end{aligned}$$

Here

$$\frac{1}{p}+\frac{1}{p'}=1. \tag{3.3}$$

Combining (3.2) with (3.3), we solve the indices $p=\frac{4}{3}$ and $p'=4$. We claim that

$$\left\| \left(\frac{1}{|x|^p} \underset{x}{*} \left\| \nabla_v f^n(x,v,s) \right\|_{L_v^1}^p \right)^{\frac{1}{p}} \right\|_{L_x^{p'}} \le C \tag{3.4}$$

uniformly in n. Thus,

$$\left\| f^{n+1}(x,v,t)-f^n(x,v,t) \right\|_{L_x^pL_v^1} \le \int_0^t \left\| f^n(\cdot,s)-f^{n-1}(\cdot,s) \right\|_{L_x^pL_v^1}. \tag{3.5}$$

By a process of induction, we obtain

$$\left\| \left(f^{n+1}-f^n\right)(x,v,t) \right\|_{L_x^pL_v^1} \le \frac{C^nt^n}{n!} \max_{t\in[0,T]} \left\| \left(f^1-f^0\right)(x,v,t) \right\|_{L_x^pL_v^1},$$

which implies f^n converges to some f in $L^\infty_{\mathrm{loc}}([0,T),L_x^pL_v^1)$.

Proof of the claim of (3.4).

A direct calculation by using the Hardy inequality yields

$$\begin{aligned} &\left\| \left(\frac{1}{|x|^p} \underset{x}{*} \left\| \nabla_v f^n(x,v,s) \right\|_{L_v^1}^p \right)^{\frac{1}{p}} \right\|_{L_x^{p'}} \\ &\quad \le C \left\| \left\| \nabla_v f^n(x,v,s) \right\|_{L_v^1}^p \right\|_{L_x^q} \\ &\quad = C \left\| \nabla_v f^n(x,v,s) \right\|_{L_x^{pq}L_v^1}, \end{aligned}$$

here $\frac{1}{q}+\frac{p}{2}=1+\frac{p}{p'}$, $q>1$.

Differentiating equation (3.1) by D_x and D_v, respectively, we get

$$\begin{cases} \frac{\partial(D_xf^{n+1})}{\partial t}+v\cdot\nabla_x(D_xf^{n+1})+E^n\cdot\nabla_v(D_xf^{n+1})-\sigma\Delta_v(D_xf^{n+1})=-\nabla_xE^n\cdot\nabla_vf^{n+1}; \\ \frac{\partial(D_vf^{n+1})}{\partial t}+v\cdot\nabla_x(D_vf^{n+1})+E^n\cdot\nabla_v(D_vf^{n+1})-\sigma\Delta_v(D_vf^{n+1})=-\nabla_xf^{n+1}. \end{cases} \tag{3.6}$$

Applying Lemma 2.3 to (3.6) yields

$$\begin{cases} \|\nabla_xf^{n+1}\|_{L_x^pL_v^1} \le \|\nabla_xf_0\|_{L_x^pL_v^1}+\int_0^t \|\nabla_xE^n\cdot\nabla_vf^{n+1}\|_{L_x^pL_v^1}\,ds; \\ \|\nabla_vf^{n+1}\|_{L_x^pL_v^1} \le \|\nabla_vf_0\|_{L_x^pL_v^1}+\int_0^t \|\nabla_xf^{n+1}\|_{L_x^pL_v^1}\,ds, \end{cases} \tag{3.7}$$

where $p>1$. Since

$$\left\| \nabla_xE^n\cdot\nabla_vf^{n+1} \right\|_{L_x^pL_v^1} \le \left\| \nabla_xE^n \right\|_{L_x^{p_1}} \left\| \nabla_vf^{n+1} \right\|_{L_x^{p_2}L_v^1}, \qquad \frac{1}{p_1}+\frac{1}{p_2}=\frac{1}{p}, \tag{3.8}$$

$$\left\| \nabla_xE^n \right\|_{L_x^{p_1}} \le \left\| \frac{1}{|x|^{N-1}} \underset{x}{*} \nabla_x\rho^n \right\|_{L_x^{p_1}} \le C \left\| \nabla_xf^n \right\|_{L_x^{pq}L_v^1}, \qquad \frac{1}{pq}+\frac{1}{2}=1+\frac{1}{p_1}. \tag{3.9}$$

We require $p_1 \le p'$, $p < p'$, this is an easy thing. Note that $L^{pq}(\Omega) \subset L^p(\Omega)$ and $L^{p_2}(\Omega) \subset L^p(\Omega)$. Plugging (3.8) and (3.9) into (3.7) yields

$$\begin{cases} \|\nabla_x f^{n+1}\|_{L^p_x L^1_v} \le \|\nabla_x f_0\|_{L^p_x L^1_v} + \int_0^t \|\nabla_x f^n\|_{L^p_x L^1_v} \|\nabla_v f^{n+1}\|_{L^p_x L^1_v}\, ds; \\ \|\nabla_v f^{n+1}\|_{L^p_x L^1_v} \le \|\nabla_v f_0\|_{L^p_x L^1_v} + \int_0^t \|\nabla_x f^{n+1}\|_{L^p_x L^1_v}\, ds. \end{cases}$$

Next, we only have to solve the Gronwall inequalities in the form

$$\begin{cases} f(t) \le C_0 + \int_0^t f(s) g(s)\, ds, \\ g(t) \le C_1 + \int_0^t f(s)\, ds. \end{cases}$$

These inequalities only hold in finite time, we denote the maximal existence time by T, *i.e.*, for $t \in [0, T)$, there exists $\alpha(t) \in L^\infty([0, T))$ such that

$$\|\nabla_v f^n(t)\|_{L^p_x L^1_v} \le \alpha(t), \qquad \|\nabla_x f^n(t)\|_{L^p_x L^1_v} \le \alpha(t).$$

Therefore, the claim of (3.4) holds.

According to the standard weak convergence process, we conclude that f is a solution of the Cauchy problem of equations of (1.1).

Step 2: Regularity of the solution.

Denote D_x or D_v by D. Since $f^n \to f$ in $L^p_x L^1_v$, which deduces $Df^n \to Df$ in $\mathcal{D}'(\Omega_x \times \mathbb{R}^2_v)$, note that $\|Df^n\|_{L^p_x L^1_v} \le \text{const}$, we have $\|Df\|_{L^p_x L^1_v} \le \|Df^n\|_{L^p_x L^1_v} \le \text{const}$.

By property (i) of Proposition A.3 in [5], we conclude that f^n is nonnegative. Moreover, $f^n \to f$, a.e. $(x, v) \in \Omega_x \times \mathbb{R}^2_v$, since $f^n \to f$ in $L^p_x L^1_v$, which implies that f is nonnegative.

Step 3: Uniqueness of the solution.

The uniqueness is a direct consequence of

$$\|(f - \tilde{f})(x, v, t)\|_{L^p_x L^1_v} \le C \int_0^t \|(f - \tilde{f})(x, v, t)\|_{L^p_x L^1_v}, \tag{3.10}$$

which in turn is a result of a very similar process to (3.5).

Next we are going to deal with the boundary, *i.e.*, to show that the solution satisfied the boundary condition.

On the one hand, by Lemma 2.3, we have $\|f^n\|_{L^1_v} \le \|f_0\|_{L^1_v}$ for x a.e.; on the other hand, $\|f^n(t, x, v)\|_{L^1_v} \to \|f(t, x, v)\|_{L^1_v}$ for x a.e. Thus $\|f(t, x, v)\|_{L^1_v} \le \|f_0\|_{L^1_v}$ for x a.e., note the assumption of $\|f_0\|_{L^1_v}|_{\partial\Omega} = 0$ in the sense of trace of L^p_x, we conclude in the sense of trace of L^p_x, since the boundary is C^1 and $\|f^n\|_{L^1_v} \in W^{1,p}_x$:

$$\|f(t, x, v)\|_{L^1_v}|_{\partial\Omega} = 0. \qquad \square$$

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

JCC is in charge of Section 2 and participated in the computation and discussion of the main theorem, Section 3. CH is responsible for the Section 3. This paper was finished under our joint efforts. We made many modifications and approve the final manuscript together. All authors read and approved the final manuscript.

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